MTH 203: Introduction to Groups and Symmetry

Homework VI

(Due 19/10/2022)

Problems for submission

1. Let G be a nontrivial group. Two elements $g, h \in G$ are said to be *conjugate in G* if there exists $x \in G$ such that $g = xhx^{-1}$. Now define a relation $\sim_c$ on G by

$$g \sim_c h \iff g \text{ and } h \text{ are conjugate.}$$

   Show that $\sim_c$ defines an equivalence relation on G.

   Each equivalence class (denoted by $[g]_c$) induced by the relation $\sim_c$ is called a *conjugacy class of G*.

2. Consider the partition of S_n into distinct conjugacy classes under the equivalence relation $\sim_c$ mentioned above.

   (a) If $\sigma \in S_n$ is an m-cycle and $\sigma' \in S_m$ is an ℓ-cycle, then show that

$$[\sigma]_c = [\sigma']_c \iff m = \ell.$$

   [Hint: Start by showing that given a permutation σ and a cycle $(i_1\, i_2\, \ldots\, i_k)$, $\sigma(i_1\, i_2\, \ldots\, i_k)\sigma^{-1} = (\sigma(i_1)\, \sigma(i_2)\, \ldots\, \sigma(i_k))$.]

   (b) Suppose that the unique cycle decomposition of a permutation $\sigma \in S_n$ is given by

$$\sigma = \sigma_1 \sigma_2 \ldots \sigma_{k_\sigma},$$

   where each σ_i is an m_i-cycle. Then, as $\sum_{i=1}^{k_\sigma} m_i = n$, this decomposition induces a *partition* P_σ of the integer n. (In Number Theory, a *partition of a positive integer* n is a way of writing n as a sum of positive integers, up to reordering of summands.) Then show that:

   (i) $o(\sigma) = \operatorname{lcm}(m_1, m_2, \ldots, m_{k_\sigma})$.

   (ii) Given two permutations $\sigma_1, \sigma_2 \in S_n$,

$$[\sigma_1]_c = [\sigma_2]_c \iff P_{\sigma_1} = P_{\sigma_2}.$$

   (c) Using (b), determine the number of distinct conjugacy classes of S_n.

Problems for practice

1. Establish the assertions in 5.1 (iii), 5.1 (viii), 5.2 (ii), 5.2 (v) - (vi), and 5.3 (iii) of the Lesson Plan.

2. Show that for $n \geq 3$, there exists a momomorphism $D_{2n} \to S_n$.

3. Show that for $n \geq 3$, there exists a momomorphism $S_n \to S_m$ for every $m > n$.

4. Show that every normal subgroup of S_n is a disjoint union of conjugacy classes.

5. For $n \geq 3$, show that the following sets of transpositions generate S_n.

   (a) The set $A = \{(i\, i+1) : 1 \leq i \leq n-1\}$.

   (b) For $1 \leq j \leq n$, the set $B_j = \{(j\, i) : 1 \leq i \leq n \text{ and } j \neq i\}$.

   [Hint: Start by showing that every other transposition (and hence every k-cycle) is a product of the transpositions in either A or the B_j. Then use the unique cycle decomposition property of a permutation.]